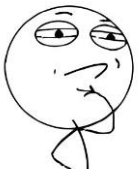
Just for your interest…

Where does e come from, and why is it so important?



$e = 2.71828\ldots$ is known as **Euler's Number**, and is considered one of the five fundamental constants in maths: $0,\ 1,\ \pi,\ e,\ i$

Its value was originally encountered by Bernoulli who was solving the following problem:
You have £1. If you put it in a bank account with 100% interest, how much do you have a year later? If the interest is split into 2 instalments of 50% interest, how much will I have? What about 3 instalments of 33.3%? And so on…

No. Instalments	Money after a year
1	$1 \times 2^1 = £2$
2	$1 \times 1.5^2 = £2.25$
3	$1 \times 1.\dot{3}^3 = £2.37$
4	$1 \times 1.25^4 = £2.44$
n	$\left(1 + \frac{1}{n}\right)^n$

As n becomes larger, the amount after a year approaches £2.71…, i.e. e!

Thus:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

But we have seen that differentiation by first principles uses 'limits'. It is therefore possible to prove from the definition above that $\frac{d}{dx}(e^x) = e^x$, and **these two definitions of e are considered to be equivalent***.

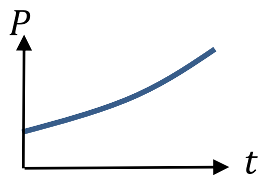
e therefore tends to arise in problems involving limits, and also therefore crops up all the time in anything involving differentiation and integration. Let's see some applications…

*You can find a full proof here in my Graph Sketching/Limits slides:
http://www.drfrostmaths.com/resources/resource.php?rid=163

Application 1: Solutions to many 'differential equations'.

Frequently in physics/maths, **the rate of change of a variable is proportional to the value itself**. So with a population P behaving in this way, if the population doubled, the rate of increase would double.

$$P \propto \frac{dP}{dt} \quad \rightarrow \quad P = k\frac{dP}{dt}$$



This is known as a 'differential equation' because the equation involves both the variable and its derivative $\frac{dP}{dt}$.

The 'solution' to a differentiation equation means to have an equation relating P and t without the $\frac{dP}{dt}$. We end up with (using Year 2 techniques):

$$\boldsymbol{P = Ae^{kt}}$$

where A and k are constants. This is expected, because we know from experience that population growth is usually exponential.

Application 2: Russian Roulette

A scene from one of Dr Frost's favourite films, *The Deer Hunter*.

I once wondered (as you do), if I was playing Russian Roulette, where you randomly rotate the barrel of a gun each time with n chambers, but with one bullet, what's the probability I'm still alive after n shots?

The probability of surviving each time is $1-\frac{1}{n}$, so the probability of surviving all n shots is $\left(1-\frac{1}{n}\right)^n$. We might consider what happens when n becomes large, i.e. $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$. In general, $e^k = \lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n$.

Thus $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e^{-1} = \frac{1}{e}$, i.e. I have a 1 in e chance of surviving. Bad odds!

This is also applicable to the lottery. If there was a 1 in 20 million chance of winning the lottery, we might naturally wonder what happens if we bought 20 million (random) lottery tickets. There's a 1 in e (roughly a third) chance of winning no money at all!

Application 3: Secret Santa

You might have encountered $n! = n\times(n-1)\times\cdots\times 2\times 1$, said "$n$ factorial" meaning "*the number of ways of arranging n objects in a line*". So if we had 3 letters ABC, we have $3! = 6$ ways of arranging them.

ABC, ACB, BAC, BCA, CAB, CBA

Meanwhile, $!n$ means the number of **derangements** of n, i.e. the arrangements where **no letter appears in its original place**.

For ABC, that only gives BCA or CAB, so $!3 = 2$. This is applicable to '**Secret Santa**' (where each person is given a name out a hat of whom to give their present to) because ideally we want the scenario where *no person gets their own name*.

Remarkably, a derangement occurs an e-th of the time. So if there are 5 people and hence $5! = 120$ possible allocations of recipient names, we only get the ideal Secret Santa situation just $\frac{120}{e} = 44.15 \rightarrow 44$ times. And so we get **my favourite result in the whole of mathematics**:

$$!n = \left[\frac{n!}{e}\right]$$

(where [...] means round)