Finding the formula for a sequence

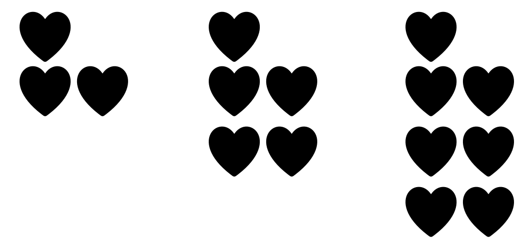
Instructions

1. For each question, there is a sequence of images. You have to <u>find a formula that connects the number in the sequence and the total number of shapes</u> (unless otherwise stated).
2. Put the formula you have found in the table provided.
3. Use your formula to <u>find the value</u> for the <u>100^{th} picture in the sequence</u>.
4. The questions are divided into 4 levels: Level 1, Level 2, Level 3 and Level ☠. The questions in the last level are more open-ended investigations, and it is unlikely you will find the full answer.
5. You can go on to the next level once you have put up your hand and shown your solutions to the teacher.

www.drfrostmaths.com

Example

n=1 n=2 n=3



"The number of hearts H"

Formula	100^{th} value
H = 2n + 1	201

We have 1 heart at the top of each image. We also have a rectangle of width 2 and height n. So we have 2n + 1 hearts in total.

H = (2 x 100) + 1
= 201

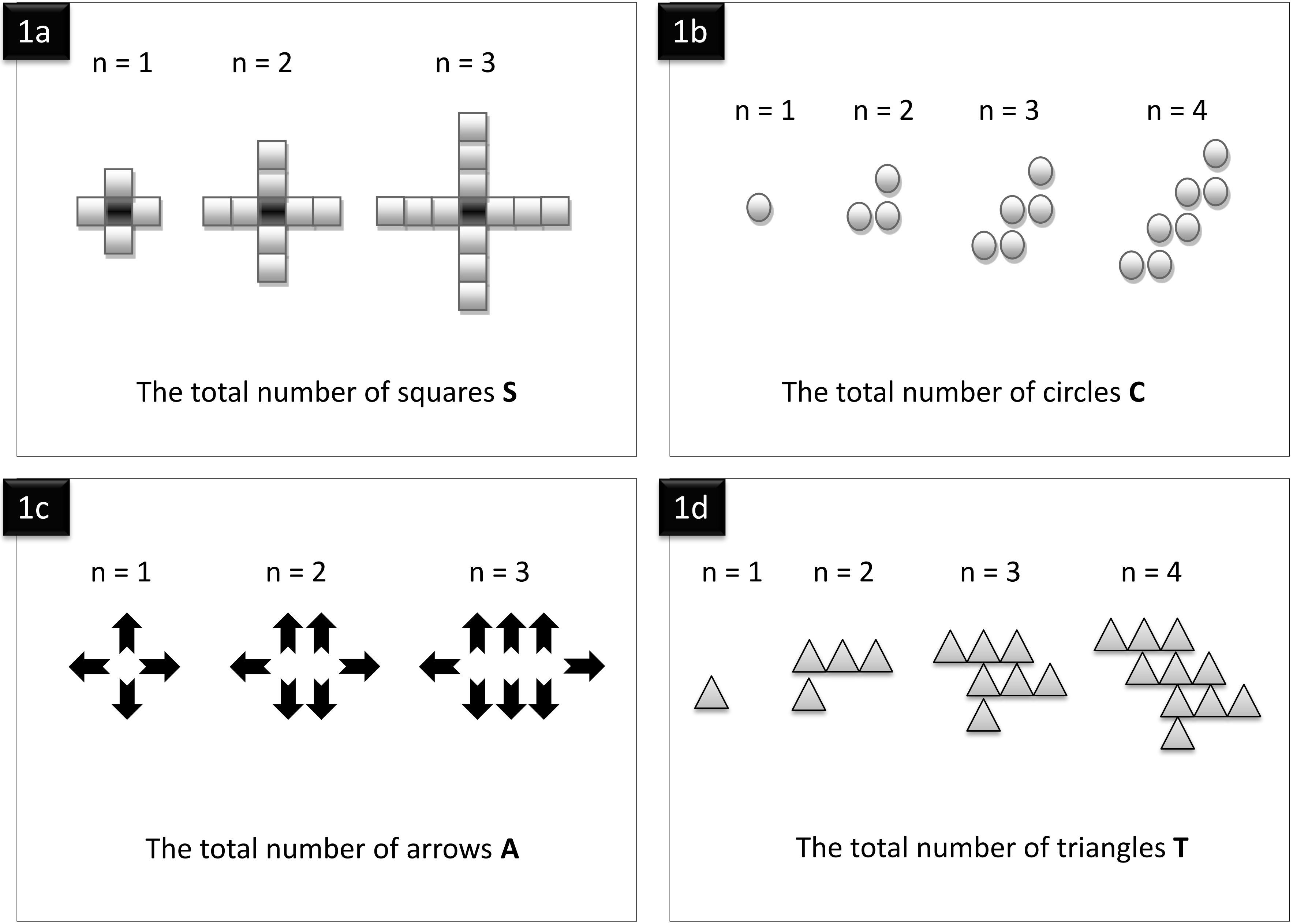
Answer Sheet

	Formula	100^{th} value
1a		
1b		
1c		
1d		
1e		
1f		

	Formula	100^{th} value
2a		
2b		
2c		
2d		
2e		
2f		

	Formula	100^{th} value
3a		
3b		

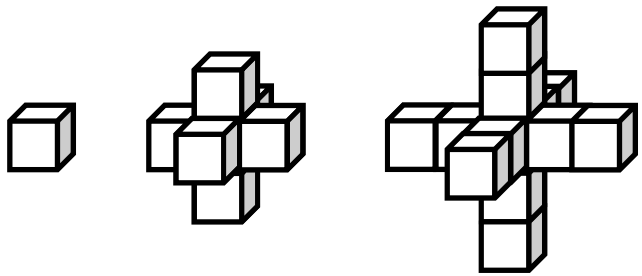
	Your Findings
☠a	
☠b	



1a
n = 1
n = 2
n = 3
The total number of squares **S**
1b
n = 1
n = 2
n = 3
n = 4
The total number of circles **C**
1c
n = 1
n = 2
n = 3
The total number of arrows **A**
1d
n = 1
n = 2
n = 3
n = 4
The total number of triangles **T**

1e

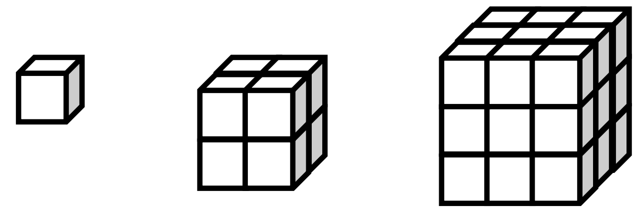
n = 1 n = 2 n = 3



The total number of cubes **C**

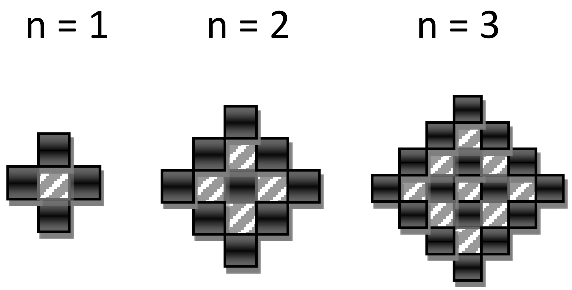
1f

n = 1 n = 2 n = 3



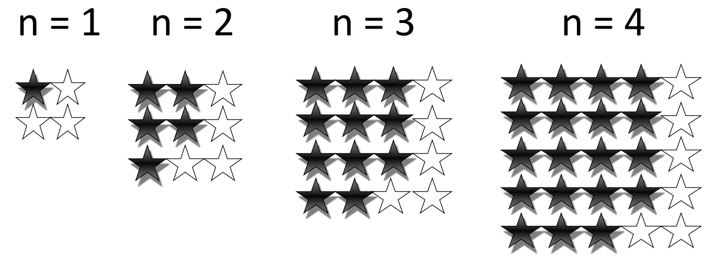
The total number of cubes **C**

2a



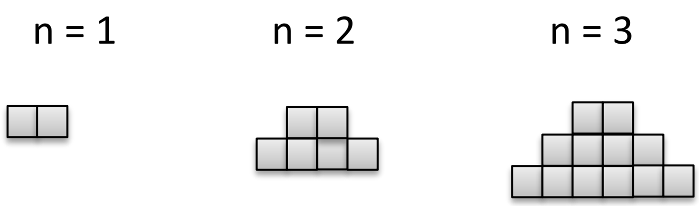
The total number of squares **S** (both black and striped)

2b



The total number of <u>black</u> stars **B**

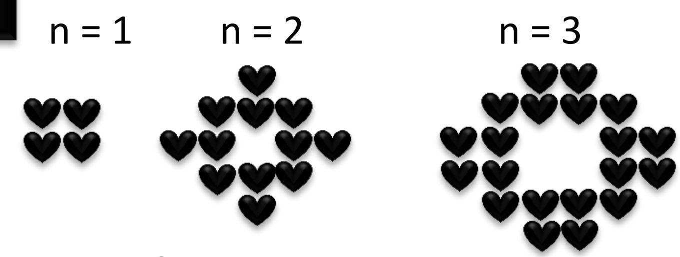
2c

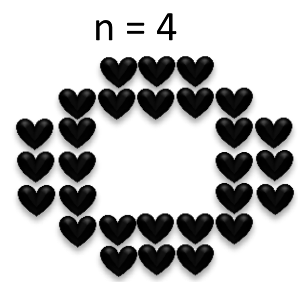


The total number of squares **S**

(hint: it might help to first find the average number of squares on each row in terms of n)

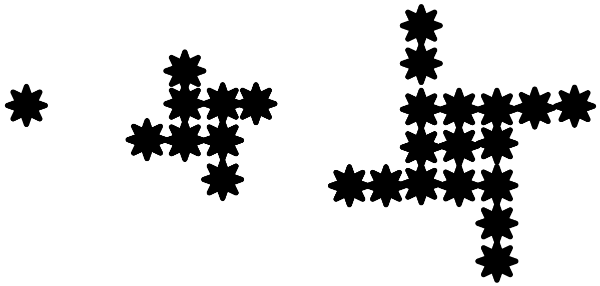
2d





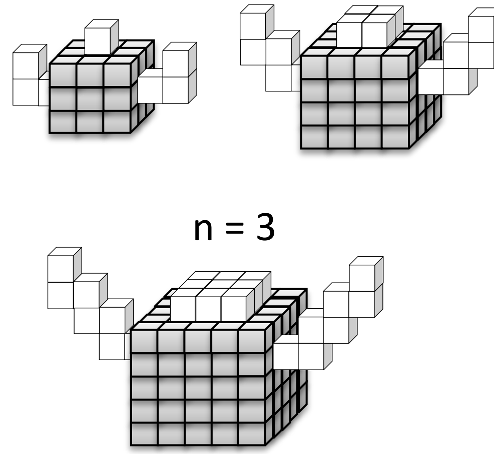
The total number of hearts **H**

2e



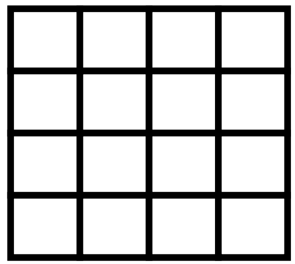
The total number of stars **S**

2f



The total number of cubes **M** in the monster (his insides are solid, not hollow)

3a



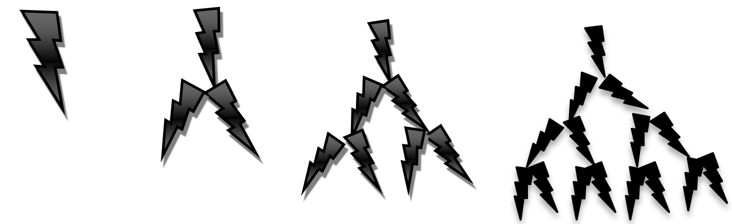
n = 4

The number of squares **S** (<u>of any size</u>) given the width of the large square **n**

(Hint: the sum of the first n square numbers is $\frac{1}{6}n(n+1)(2n+1)$)

3b

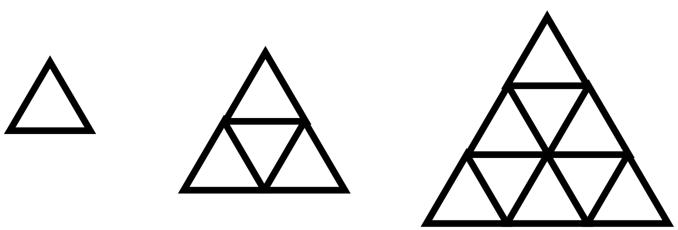
n = 1 n=2 n=3 n=4



The number of lightning bolts **B** based on the intensity of the lightning **n**.

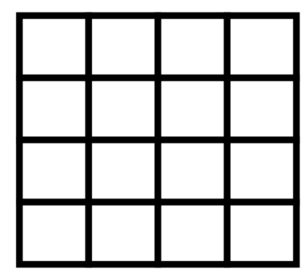
☠a

n = 1 n=2 n=3



The number of triangles **T** (<u>of any size</u>)

☠b



n = 4

The number of <u>rectangles</u> **R** (<u>of any size</u>) given the width of the large square **n**

Answers (left unsimplified)

	Formula	100^{th} value
1a	$S = 4n + 1$	401
1b	$C = 2n - 1$	199
1c	$A = 2n + 2$	202
1d	$T = 3n - 2$	298
1e	$C = 6n - 5$	595
1f	$C = n^3$	1,000,000

	Formula	100^{th} value
2a	$S = n^2 + (n+1)^2$	20201
2b	$B = n^2 + (n-1)$	10099
2c	$S = n(n+1)$	10100
2d	$H = 8n - 4$	796
2e	$S = n^2 + 4(n-1)$	10396
2f	$M = (n+2)^3 + n^2 + 2(2n + 1)$	1,071,610

	Formula	100^{th} value
3a	$\frac{1}{6}n(n+1)(2n+1)$	338,350
3b	$B = 2^n - 1$	1.27×10^{30}

	Your Findings
☠a	See overleaf
☠b	See overleaf

☠ Answers

☠a

The best strategy here is to split triangles into those pointing up, and those pointing down, and to then spot patterns for triangles of each type and of each size for different n.

Pointing Up: The number of triangles of size 1 is the nth triangular number. Similarly the number of triangles of size 2 is the (n-1)th triangular number. We find therefore that the total number of up triangles is the sum of the first n triangular numbers. This is given by the formula:

$$\frac{1}{6}n(n+1)(n+2)$$

Pointing Down: Here it gets more tricky, as you'll find a new larger triangle only appears for even n.

For odd n, our sums are 0, 3, 10+3, 21+10+3 for n=1,3,5,7 respectively. This again is a sum of triangular numbers, but skipping every other number in the sequence.

Sums of numbers generated by a quadratic formula can be enumerated by a cubic formula, so I found the coefficients $an^3 + bn^2 + cn + d$ by using the first few values in the sequence. But 0, 3, 13, 34 are the 1st, 3rd, 5th and 7th values in the sequence, so I used $n = 2k - 1$, then formed four simultaneous equations using $ak^3 + bk^2 + ck + d = N_k$, where N_1 is 0, N_2 is 3, etc. before substituting k for (n+1)/2 in the resulting formula.

☠ Answers

☠a

(Continued) We can use a similar approach to get a formula for down triangles when n is even.

The final solution (after simplification) is:

$$T = \frac{\left|\cos\left(\frac{\pi n}{2}\right)\right| n(n+2)(2n+1) + \left|\sin\left(\frac{\pi n}{2}\right)\right| (n(n+2)(2n+1) - 1)}{8}$$

The trigonometric terms yield 1 and 0 for even n, and 0 and 1 for odd n, allowing you to combine the formulas for odd and even n. So for even n, this simplifies to <u>n(n+2)(2n+1) / 8</u>, and for odd n, to <u>(n(n+2)(2n+1) - 1) / 8</u>.

☠ Answers

☠b

One strategy here is to consider rectangles of each size, say width w and height h. Students might be able to work out the number of rectangles N_{wh} as:

$$N_{wh} = (n - w + 1) \times (n - h + 1)$$

The total number of rectangles then is all the rectangles of width 1 to n and height 1 to n:

$$\begin{aligned}
&\sum_{w=1}^{n} \sum_{h=1}^{n} (n - w + 1) \times (n - h + 1) \\
&= \sum_{w=1}^{n} \sum_{h=1}^{n} wh \\
&= \sum_{w=1}^{n} \left[w \sum_{h=1}^{n} h \right] \\
&= \left[\sum_{h=1}^{n} h \right] \left[\sum_{w=1}^{n} w \right] \\
&= \left[\frac{n(n+1)}{2} \right]^2 = \frac{n^2(n+1)^2}{4}
\end{aligned}$$

Since the inner sum doesn't depend on w, so we can factor out.

Again factor by the same reasoning, so we now have to product of two easy summations.

This is a standard formula for the sum of 1 to n.